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A Global Optimization Approach to a Water Distribution Network Design Problem

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Abstract. In this paper, we address a global optimization approach to a water distribution network design problem. Traditionally, a variety of local optimization schemes have been developed for such problems, each new method discovering improved solutions for some standard test problems, with no known lower bound to test the quality of the solutions obtained. A notable exception is a recent paper by Eiger *et al.* (1994) who present a first global optimization approach for a loop and path-based formulation of this problem, using a semi-infinite linear program to derive lower bounds. In contrast, we employ an arc-based formulation that is linear except for certain complicating head-loss constraints and develop a first global optimization scheme for this model. Our lower bounds are derived through the design of a suitable Reformulation-Linearization Technique (RLT) that constructs a tight linear programming relaxation for the given problem, and this is embedded within a branch-and-bound algorithm. Convergence to an optimal solution is induced by coordinating this process with an appropriate partitioning scheme. Some preliminary computational experience is provided on two versions of a particular standard test problem for the literature for which an even further improved solution is discovered, but one that is verified for the first time to be an optimum (within $1 of cost), without any assumed *a priori* bounds on the flows. Two other variants of this problem are also solved exactly for illustrative purposes and to provide researchers with additional test cases having known optimal solutions. Suggestions on a more elaborate study involving several algorithmic enhancements are presented for future research.

Key words: Water distribution systems, network design, reformulation-linearization technique (RLT).

1. Introduction

The problem of designing a reliable and cost effective **Water Distribution System (WDS)** is of considerable importance because of the strong dependence of society on this natural resource. In the light of aging, deteriorating, water distribution systems in many cities throughout the world, this problem is becoming one of increasing global importance. In a recent international conference, "Integrated Computer Applications for Water Supply and Distribution", held in Leicester, UK, September 7–9, 1993, practitioners, consultants, city engineers, and academic researchers discussed and exposed the need for designing comprehensive models that integrate network reliability and redundancy issues, network expansion and pipe sizing decisions, and multi-period economic analysis, all within a holistic framework. A companion paper by Sherali *et al.* (1996a) discusses such an approach, and presents a single-stage network design model as a key subproblem that needs to be solved within an overall design loop. This particular problem is a hard, nonconvex, problem that has been widely researched over two decades now, and the literature contains a host of local optimization schemes that have yielded better and better approximate solutions for various standard test problems. However, these methods lack even the determination of an adequate lower bound that might assist in evaluating a local optimum. The present paper describes a first global optimization algorithm to solve this particular model. (See below for a discussion on one other recent global optimization procedure due to Eiger *et al.* (1994), that has been applied to an alternative model formulation of this problem.)

The most general WDS problem is to modify and/or expand the design of an existing network so that it is capable of satisfying the varied anticipated demand patterns for water at required pressure levels, even while experiencing breakages in the network. If the network is designed with low energy heads and using undersized or rough pipes in a skeletal fashion, then flow and/or pressure requirements will not be met during certain demand peaks or under various pipe failure scenarios. On the other hand, if the energy sources and pipes are overdesigned, or if there are too many redundant paths, then increased costs may lead to an inefficient solution. Therefore, the problem at hand requires a cost effective network design and replacement strategy that satisfies stated hydraulic requirements under various likely demand patterns and failure modes. As mentioned above, Sherali *et al.* (1996a) have recently developed an integrated pipe-reliability-and-cost, and network-optimization approach that analyzes pipe reliability and annualized maintenance costs, along with replacement recommendations, as well as the design and sizing of expanded and replaced sections of the network. The core driver in such a design approach is a *single-stage network optimization model* that determines a least cost pipe sizing (diameters and lengths), and energy requirements (elevated source head levels), for a fixed demand pattern. The problem formulated is a nonlinear program to minimize the cost of designing pipes and elevating energy heads of sources subject to satisfying hydraulic flow and pressure requirements at the different nodes of the distribution network. Pipe links are limited to be composed of existing pipe segments in the network, and new pipe segments selected from commercially available pipe diameters. Existing pipe links may be retained intact, or may be replaced either partially (segment-wise) or completely. This problem turns out to be a hard nonconvex optimization problem that has many local optima, different from a globally minimum cost design, and has hence proven to be difficult to solve.

One of the first novel optimization approaches to solve this problem was due to Alperovits and Shamir (1977) who proposed the popular (successive) linear programming gradient (LPG) method. Since for a fixed set of flows, the problem reduces to a linear program in the pressure heads and pipe lengths, the authors suggest a procedure that projects the problem onto the space of the flow variables, and heuristically adjust these variables based on variations in the optimal value of the associated linear programming problem. Quindry *et al.* (1979) showed that Alperovits and Shamir had missed certain terms in their gradient expressions. However, the inclusion of these terms rendered the procedure ineffective for large-scale problems. Later, Quindry *et al.* (1981), Fujiwara *et al.* (1987), Kessler and Shamir (1989), and Fujiwara and Khang (1990) proposed alternative derivations of the linear programming based gradient expressions along with other algorithmic enhancements to improve the computational efficiency of the LPG approach.

Lansey and Mays (1985) simplified the solution of this nonlinear single-stage model by incorporating a network simulator within the optimization model, and by applying general reduced-gradient and optimal control theory concepts. On the other hand, Gessler (1985) and Loubser and Gessler (1993) simplified this problem by using an enumeration approach that examines all possible combinations of decision options, testing each for feasibility and cost. The obvious combinatorial explosion of such an approach is somewhat mitigated by grouping together similar classes of pipes and by pruning infeasible or inferior combinations that might be evident from among the candidate list of solutions.

An alternative decomposition/projection algorithm was proposed by Sherali and Smith (1993), where the problem is projected onto the space of the network design variables (pipe lengths of various diameters and source elevation heads), and an auxiliary convex cost network flow subproblem of the type analyzed by Collins *et al.* (1978) is used to guide the variations in the design variables.

Several other refinements have been proposed in the design of water distribution systems. Morgan and Goulter (1985) suggest an enhancement in the process of designing or expanding a looped WDS by treating multiple demand loads and pipe failure scenarios, along with network connectivity issues in order to achieve adequate hydraulic redundancy. Hobbs and Hepenstal (1989) suggest another enhancement based on using Monte-Carlo simulation techniques to assess the network performance under various random operating conditions within the design process. Other papers attempt to integrate reliability issues more intimately within the network design process in an iterative, multi-stage process. Examples of such approaches include the works of Rowell and Barnes (1982), Loganathan *et al.* (1990), and Fujiwara and Tung (1991). Walski (1984, 1985) also provides a discussion on related real-world considerations in the design of water distribution systems, and Sherali *et al.* (1996a) discuss how the various models developed in the literature might be integrated to jointly address several such issues.

All of the above procedures employ heuristics for the single-stage network optimization problem that, at best, might converge to locally optimal solutions. A first global optimization approach has been recently proposed by Eiger *et al.* (1994). This approach is applied to an alternative formulation of the single-stage design problem than the one considered herein, in that it enforces the hydraulic consistency requirements via an enumeration of all possible basic loops and source-to-demand node paths in the network, as opposed to employing a link-wise formulation of these constraints. A branch-and-bound algorithm is proposed based on partitioning the hyperrectangle restricting the flows into several subrectangles (a precise scheme is unspecified). At each node of the branch-and-bound tree, a subgradient-based heuristic is applied to determine an upper bound via the nonsmooth, nonconvex, projection of the problem onto the space of the flow variables. A different relaxed, duality-based, linear programming formulation is used to compute lower bounds. Although some promising results are presented, the procedure appears to experience convergence difficulties which necessitates the use of highly restrictive initial bounding hyperrectangles. Sherali *et al.* (1996b) provide some insights into the derivation of Eiger *et al.*'s lower bound, exhibiting why it might tend to be weak, and suggesting enhancements to tighten this bound.

The present paper addresses a different global optimization approach applied to an alternative formulation of the WDS design problem. The proposed algorithm is also a branch-and-bound procedure, but one that uses a Reformulation-Linearization Technique (RLT) in order to compute linear programming based lower bounds, and coordinates this with a suitable partitioning scheme in order to induce convergence to a global optimum. Upper bounds are computed by applying a simple heuristic to the lower bounding solution. We apply this algorithm to two variants of the classic Alperovits and Shamir (1977) test problem that have been considered in the literature, and discover improved solutions over the best known solutions for these problems, verifying the true optimality of these solutions for the first time ever to within a \$1 difference between the global lower and upper bounds, without assuming any *a priori* bounds on the flows. Two other variants of this problem are also solved to exact global optimality to illustrate the performance of the algorithm and to provide additional test cases. The purpose of this paper, therefore, is to present to the global optimization community a particular model for this important problem of designing water distribution systems, to describe the principal framework of an approach to solve this model, and to provide some test cases with known optimal solutions, without having assumed any *a priori* bounds on the flow variables. Although we demonstrate some encouraging results as indicated above, we also present several algorithmic refinements and enhancements that need to be made in order to make this algorithm practical and effective for large-scale problems. Such refinements along with more extensive computations will be pursued in future research.

The remainder of this paper is organized as follows. Section 2 presents the model formulation and Section 3 discusses the design of an RLT-based lower bounding the problem and presents its fundamental theoretical properties. This construct is embedded within an infinitely convergent branch-and-bound algorithm that is described in Section 4. Section 5 presents our preliminary computational experience, and Section 6 concludes the paper with a discussion on possible algorithmic enhancements.

2. Model Formulation

The single-stage pipe network design problem seeks to determine a least cost design of a given network configuration in order to satisfy a specified anticipated demand at acceptable pressure head levels. The network requires that various *links* connecting pairs of designated nodes be comprised of possibly different *segments*, each segment being a collection of standard, commercially available, uniform pipe *sections* having specific values of diameter, roughness (Hazen–Williams) coefficient, and associated cost. Each link may have existing segments of specified lengths and age that are to be retained in the design. (Sherali *et al.* (1996a) describe an analytical technique for *a priori* making the decision to retain or replace existing pipe segments.) Furthermore, the design requires the determination of any additional head elevation (for example, via pumps) that should be provided, at an accompanying cost, at the reservoirs/sources nodes. This additional head (denoted by the decision variables H_{si} below) might be required depending on the available reservoir head, the required pressure heads at the demand nodes, and the frictional head losses in the connecting pipes. The design should be such that it is capable of sustaining flows that satisfy supply and demand flow restrictions at the various network nodes, and such that the resulting pressure heads due to frictional head loss relationship is a nonlinear, nonconvex, function of the flow, the pipe length, and the diameter, and is the feature that renders this problem hard to solve.

To present a mathematical model for this problem, consider the following notation. (In general, for any subscripted notation defined below, the same symbol without subscripts will be used to denote a vector of the corresponding entities.)

$N = \{1, \ldots, n\}$:	set of nodes in the network.
$S \subset N, D \equiv N - S$:	set of source and demand nodes, respectively.
b_i:	net water supply or demand rate (m^3/hr) corresponding to node $i \in N$. (By convention, this is taken as positive for source nodes and negative for demand nodes. Also, $\sum_{i \in N} b_i$ is assumed to be nonnegative.)
E_i:	ground elevation (m) of node i.
H_i:	*decision variable* representing the established head (m) at node i, above the level E_i.
F_i:	available fixed energy head (m) at source node $i \in S$.
H_{si}:	*decision variable* representing the additional head elevation (m) provided at source node $i \in S$.
H_{si}^U:	a practical upper bound imposed on the head elevation H_{si} for $i \in S$.

c_{si}: annualized cost per unit energy head (*$/m/yr*) provided at source node $i \in S$.

$[H_{iL}, H_{iU}]$: acceptable interval for the energy head (m) at demand node $i \in D$.

A: set of directed arcs or links (i, j) and (j, i) for each connected pair of nodes i and j in the given network configuration. (In practice, we can assume that only unidirectional links are incident at source nodes.)

Q_{ij}: *decision variable* representing the flow rate (m^3/hr) on link $(i, j) \in A$.

Q_{ij}^L, Q_{ij}^U: specified lower and upper bounds on Q_{ij}, respectively, based on some preprocessing logical or optimality analysis (see Remark 2 below).

$\Omega = \{Q : Q^L \leq Q \leq Q^U\}$: Hyperrectangle restricting the flows. (Each branch-and-bound node will principally differ in the specification of Ω.)

$L_{ij} (= L_{ji})$: pipe length (m) corresponding to link (i, j) (or $(j, i)) \in A$.

$\{d_k, k = 1, \ldots, K\}$: set of standard available pipe diameters (inches).

X_{ijk}: length (m) of existing segments of link $(i, j) \in A$ having a diameter d_k that is selected for continued use. (Note that $X_{ijk} \equiv X_{jik}$.)

x_{ijk}: *decision variable* representing the length (m) of a new segment of link $(i, j) \in A$ that is to be constructed, having a diameter d_k. (Note that $x_{ijk} \equiv x_{jik}$ and so, only one of each pair of variables should be used in the formulation by appropriate substitution.)

c_{ijk}: annualized construction and maintenance cost per unit length ($/m/yr) of link $(i, j) \in A$ that has a diameter d_k.

C_{HWN}: assumed Hazen–Williams coefficient for new pipes for computing frictional head losses.

$C_{HWE(i,j,k)}$: assumed Hazen–Williams coefficient for the existing pipe segment of length X_{ijk}, corresponding to link $(i, j) \in A$ of diameter d_k.

$$\phi_{ij}(Q_{ij}, x_{ij.}, X_{ij.}) \;=\; (1.52)10^4 Q_{ij}^{1.852} \sum_{k=1}^{K}$$

$$\times \left[\frac{x_{ijk}}{d_k^{4.87} C_{HWN}^{1.852}} + \frac{X_{ijk}}{d_k^{4.87} C_{HWE(i,j,k)}^{1.852}} \right] : \tag{1a}$$

pressure head loss (m) due to friction in link (i,j), where $x_{ij.} \equiv (x_{ijk}, k = 1, \ldots, K)$ and $X_{ij.} \equiv (X_{ijk}, k = 1, \ldots, K)$.

REMARK 1 (*Frictional Head Loss Relationship).* The frictional head loss expression for $\phi_{ij}(Q_{ij}, x_{ij}, X_{ij.})$ given in (1a) is the one that is most commonly used in the literature, as well as in all standard test problems, and corresponds to smooth flow conditions (Walski, 1984). However, Walski (1984) also suggests an alternative expression that corresponds to rough flow conditions that has an exponent of 2 for the flow variable Q_{ij}. This expression is given as follows, where $\tilde{Q}_{ij}$ is the (fixed) flow at which the Hazen–Williams coefficient was measured.

$$\begin{aligned} \phi_{ij}(Q_{ij}, x_{ij.}, X_{ij.}) \;=\; & (1.52)10^4 \frac{Q_{ij}^2}{\tilde{Q}_{ij}^{0.148}} \sum_{k=1}^{K} \\ & \times \left[\frac{x_{ijk}}{d_k^{4.87} C_{HWN}^{1.852}} + \frac{X_{ijk}}{d_k^{4.87} C_{HWE(i,j,k)}^{1.852}} \right] \end{aligned} \tag{1b}$$

In either case, for developing our model and algorithm, we will denote (1a) or (1b) as given by (1c) below, for some exponent e on the flow variable Q_{ij}, and we will indicate which specific expression is being used in our computational results presented later in Section 5.

$$\phi_{ij}(Q_{ij}, x_{ij.}, X_{ij.}) = Q_{ij}^e \ell(x, X)_{ij}. \tag{1c}$$

□

The network optimization problem (**NOP**), restricted on Ω, can then be formulated as follows:

$$\textbf{NOP } (\Omega) : \quad \underline{\text{Minimize}} \quad \sum_{\substack{(i,j)\in A \\ i<j}} \sum_{k=1}^{K} c_{ijk} x_{ijk} + \sum_{i \in S} c_{si} H_{si} \tag{2a}$$

$$\underline{\text{subject to}} \quad \sum_{j:(i,j)\in A} Q_{ij} - \sum_{j:(j,i)\in A} Q_{ji} \leq b_i \text{ for each } i \in S \tag{2b}$$

$$\sum_{j:(i,j)\in A} Q_{ij} - \sum_{j:(j,i)\in A} Q_{ji} = b_i \text{ for each } i \in D \tag{2c}$$

$$(H_i + E_i) - (H_j + E_j) = \\ = \begin{cases} \phi_{ij}(Q_{ij}, x_{ij\cdot}, X_{ij\cdot}) & \text{if } Q_{ij} > 0 \\ \leq 0 & \text{if } Q_{ij} = 0 \end{cases} \quad \text{for each } (i,j) \in A \tag{2d}$$

$$H_i + E_i \leq F_i + H_{si} \quad \text{for each } i \in S \tag{2e}$$

$$H_{iL} \leq H_i + E_i \leq H_{iU} \quad \text{for each } i \in D \tag{2f}$$

$$\sum_{k=1}^{K} (x_{ijk} + X_{ijk}) = L_{ij} \quad \text{for each } (i,j) \in A, i < j \tag{2g}$$

$$Q_{ij}^L \leq Q_{ij} \leq Q_{ij}^U \quad \forall (i,j) \in A, 0 \leq H_{si} \leq H_{si}^U \quad \forall i \in S, \\ x_{ijk} \geq 0, \quad \forall (i,j) \in A, i < j, \quad k = 1, \ldots, K. \tag{2h}$$

The objective function, Equation (2a), in the above model denotes the total annualized construction plus maintenance costs. The constraints (2b) and (2c) enforce the conservation or continuity of flow at each node in the network. The constraints (2d) represent the conservation of energy or head loss constraints for each pipe in the direction of positive flow. Note that these constraints imply that $H_i + E_i > H_j + E_j$ whenever $Q_{ij} > 0$, and so, we will never have both Q_{ij} and Q_{ji} positive in any feasible solution. The constraints (2e) represent the head available at each source node $i \in S$, constraints (2f) represent bounds on the head levels enforced at each demand note, and constraints (2g) establish the appropriate constructed pipe link lengths. Finally, constraints (2h) represent logical or bound restrictions.

REMARK 2 (*Determining Bounds on Flows).* Note that the bounds on the flows can be determined by practical considerations based on the network configuration and the supply and demand rate information. Although any implied bounds on the flow variables would suffice in theory, from the viewpoint of computational effectiveness, it is preferable to ascertain as tight bounds as possible using some preprocessing of data or optimality considerations, perhaps also identifying links whose flow direction is evident by the nature of the particular problem. (Also, see Eiger *et al.* (1994, p. 2641) for further comments on this issue.) In contrast with previous approaches, we determine provably valid initial bounds on the flow variables for use in our model, without making any *a priori* (heuristic) assumption on these values. First, we compute lower and upper bounds on the flows based on a simple logical analysis using supplies, demands, and continuity of flow over the network. Next, using a relaxation of the constraints (we employed the linear programming RLT relaxation proposed below), we sequentially minimize and maximize the flow in each link subject to this relaxed set of constraints, but with the additional constraint that the objective function (2a) be less than or equal to the best known incumbent value. (This value can be initialized at some implied maximum possible level to begin with, and can be updated whenever a new incumbent value is discovered in this process.) Note that although the revised bounds deduced for each link in this manner can be used to accordingly tighten the relaxation before solving the next pair of problems for a subsequent link, we simply performed a single independent pass through all the links that define A in the problem in order to determine the initial set of bounds Q^L and Q^U to be used in formulating (2h). □

FORMULATION OF HEAD-LOSS CONSTRAINTS (2d)

In order to mathematically restate constraints (2d), let us partition the set of arcs A into the following three disjoint subsets, depending on an admissible set of flow bounding restrictions designated by Ω. (This will hold for any node in the developed branch-and-bound enumeration tree described later in Section 4.)

$A_z =$ $\{(i,j) \in A : Q_{ij}^L = Q_{ij}^U = 0\}$. These arcs are designated to carry no flow and can be eliminated from the network.

$A_1 =$ $\{(i,j) \in A : Q_{ij}^U > 0$ *and* $(j,i) \in A_z\}$. These arcs correspond to *unidirectional links* for which the possible flow direction has been determined.

$A_2 =$ $\{(i,j)$ and $(j,i) \in A : Q_{ij}^L = 0, Q_{ij}^U > 0$, *and* $Q_{ji}^L = 0, Q_{ji}^U > 0\}$. These arcs correspond to *bidirectional links* for which the flow direction has not as yet been determined.

$A_F = A_1 \cup A_2 \equiv A - A_z =$ set of network arcs that have a possibility of nonzero flow.

Note that A can essentially be replaced by A_F in the model NOP(Ω). Now, noting the form of ϕ_{ij} in (1c), we can replace (2d) by the following set of equivalent inequalities for all $(i,j) \in A_F$.

$$(H_i + E_i) - (H_j + E_j) = [Q_{ij}^U]^e \ell(x,X)_{ij} \quad \forall (i,j) \in A_1 \ni Q_{ij}^L = Q_{ij}^U \tag{3a}$$

$$(H_i + E_i) - (H_j + E_j) = Q_{ij}^e \ell(x,X)_{ij} \quad \forall (i,j) \in A_1 \ni Q_{ij}^L < Q_{ij}^U \tag{3b}$$

$$\{(H_i + E_i) - (H_j + E_j)\} \le Q_{ij}^e \ell(x,X)_{ij} \quad \forall (i,j) \in A_2, \text{ and} \tag{3c}$$

$$Q_{ij}\{(H_i + E_i) - (H_j + E_j)\} \ge Q_{ij}^{(e+1)} \ell(x,X)_{ij} \quad \forall (i,j) \in A_2. \tag{3d}$$

It should be evident from the formulation of (3a, b) versus (3c, d) that the representation of constraints (2d) is greatly simplified once the direction of flow on the network links is determined. As will be seen in the next section, the knowledge of flow directions can also be exploited while generating the lower bounding problems in order to derive potentially tighter representations. We now proceed to develop this linear programming relaxation of problem NOP(Ω) that we will employ as the principal component in our proposed branch-and-boundalgorithm.

3. Design of a Reformulation-Linearization Technique Based Relaxation

In this section, we develop a relaxation based on the Reformulation-Linearization Technique (RLT) as propounded by Sherali and Tuncbilek (1992) and by Sherali (1996) for general polynomial programming problems, by specializing this technique, with some modifications, for Problem NOP(Ω). The resulting linear programming relaxation provides an outer-approximation to the underlying nonconvex problem. This approximation is embedded within a branch-and-bound search procedure in the following section, that is proven to converge to a global ε-optimal solution in finite time, for any specified accuracy tolerance $\varepsilon > 0$.

The fundamental idea behind the RLT strategy is to linearize nonlinear polynomial terms by substituting linear variables in their stead. However, in order to establish the required relationships between the nonlinear and the linearized terms, various valid or implied constraints need to be generated and added to the resulting relaxed linear program, **RLT**[**NOP**(Ω)]. These RLT constraints, as they are called, also serve to tighten the linear programming relaxation, and their choice and design constitutes the principal step of applying the Reformulation-Linearization Technique. In addition, the RLT design must be coordinated with an appropriate partitioning scheme that is applied to the variable bounding intervals. This must be done in a manner so that as these intervals are further and further restricted, the corresponding relaxed linear programs become tighter and tighter, inducing an infinite convergence process to an optimal solution to the original nonlinear program NOP, without the intervals necessarily approaching zero. In other words, we must ensure that along *any* infinite branch of the accompanying branch-and-bound scheme, any accumulation point of the corresponding sequence of solutions generated for the linear programming relaxations solves Problem NOP.

In our particular implementation of RLT, in order to handle general rational exponents e on the flow variables Q_{ij} in (1c) as in Sherali (1996), we will relax the constraints by using concave envelopes (chord approximations) and first-order tangential approximations to obtain upper and lower bounding affine functions for the convex functions Q_{ij}^e or $Q_{ij}^{(e+1)}$ in (3). This will yield valid inequalities having *integer* exponents to which RLT can be suitably applied. First in (3b) and (3c), since Q_{ij}^e is a convex function defined over $[Q_{ij}^L, Q_{ij}^U]$, it is bounded above by the *affine* chord $g_{ij}(Q_{ij})$ on this interval, which happens to define its concave envelope over this interval. Hence, we get, noting that $A_F \equiv A_1 \cup A_2$,

$$Q_{ij}^e \leq g_{ij}(Q_{ij}) \equiv \frac{[Q_{ij}^U]^e(Q_{ij} - Q_{ij}^L) + [Q_{ij}^L]^e(Q_{ij}^U - Q_{ij})}{(Q_{ij}^U - Q_{ij}^L)} \quad \forall (i,j) \in A_F \ni Q_{ij}^L < Q_{ij}^U. \tag{4a}$$

Furthermore, in (3b), since the convex function Q_{ij}^e lies above the first-order tangential *affine* approximation $f_{ij}(Q_{ij};\bar{Q}_{ij})$ defined at any point $\bar{Q}_{ij} \equiv [Q_{ij}^L, Q_{ij}^U]$, we get

$$Q_{ij}^e \geq f_{ij}(Q_{ij};\bar{Q}_{ij}) \equiv [e\bar{Q}_{ij}^{(e-1)}]Q_{ij} - (e-1)\bar{Q}_{ij}^e \quad \forall (i,j) \in A_1 \ni Q_{ij}^L < Q_{ij}^U. \tag{4b}$$

Symmetric to (4b), for the convex function $Q_{ij}^{(e+1)}$ in (3d), we obtain

$$Q_{ij}^{(e+1)} \geq f_{ij}(Q_{ij};\bar{Q}_{ij}) \equiv [(e+1)\bar{Q}_{ij}^e]Q_{ij} - e\bar{Q}_{ij}^{(e+1)} \quad \forall (i,j) \in A_2. \tag{4c}$$

Using (4a) in (3b) and (3c), using (4b) in (3b), and using (4c) in (3d), we relax the constraints (3) to obtain the following set of restrictions. Here, note that in applying (4b) and (4c), we have employed $\bar{Q}_{ij}$ as six particular convex combinations of Q_{ij}^L and Q_{ij}^U, including the end points, and that f_{ij} is given by (4b) and (4c) for $(i,j) \in A_1$ and A_2, respectively.

$$(H_i + E_i) - (H_j + E_j) = [Q_{ij}^U]^e \ell(x,X)_{ij} \quad \forall (i,j) \in A_1 \ni Q_{ij}^L = Q_{ij}^U \tag{5a}$$

$$\{(H_i + E_i) - (H_j + E_j)\} \leq g_{ij}(Q_{ij})\ell(x,X)_{ij} \quad \forall (i,j) \in A_F \ni Q_{ij}^L < Q_{ij}^U \tag{5b}$$

and for each $\bar{Q}_{ij} = \alpha Q_{ij}^L + (1-\alpha)Q_{ij}^U$ for $\alpha = 0$, 0.25, 0.5, 0.75, 0.875, and 1.0, we have

$$\{(H_i + E_i) - (H_j + E_j)\} \geq f_{ij}(Q_{ij};\bar{Q}_{ij})\ell(x,X)_{ij} \quad \forall (i,j) \in A_1 \ni Q_{ij}^L < Q_{ij}^U \tag{5c}$$

and

$$Q_{ij}\{(H_i + E_i) - (H_j + E_j)\} \geq f_{ij}(Q_{ij};\bar{Q}_{ij})\ell(x,X)_{ij} \quad \forall (i,j) \in A_2. \tag{5d}$$

The RLT process then operates in the following two phases.

Reformulation Phase

In this phase, we add several valid or implied nonlinear constraints to the program NOP(Ω) to capture useful relationships between the inherent polynomial terms and the linear variables that will be used to represent them, while relaxing (2d) to the constraints (5) stated above. These new restrictions are formed by generating suitable product constraints that are implied by the original problem constraints, where the product operation is designated by (*) below.

The resulting reformulated problem, denoted **RNOP**(Ω) is summarized below:

$$\begin{aligned}
&\mathbf{RNOP}(\Omega): \quad \underline{\text{Minimize}} && [\text{Objective (2a)}] && \text{(6a)}\\
& \quad \underline{\text{subject to}} && \text{(2b), (2c)} && \text{(6b)}\\
& && \text{(5a), (5b), (5c), (5d)} && \text{(6c)}\\
& && \text{(2e), (2f), (2g)} && \text{(6d)}\\
& && \text{(2h)} && \text{(6e)}\\
& && \text{along with the RLT constraints:} && \\
& && \text{(2g)} * Q_{ij} \quad \forall (i,j) \in A_F && \text{(6f)}
\end{aligned}$$

$$[\text{(2f) written for node } i] *Q_{ij} \text{ and } [\text{(2f) written for node } j] *Q_{ij} \quad \forall (i,j) \in A_2 \tag{6g}$$

$$[\text{(2f) written for node } i] *[Q_{ij}^U - Q_{ij}] \text{ and } [\text{(2f) written for node } j] *[Q_{ij}^U - Q_{ij}] \quad \forall (i,j) \in A_2 \tag{6h}$$

$$x_{ijk}(Q_{ij} - Q_{ij}^L) \geq 0 \text{ and } x_{ijk}(Q_{ij}^U - Q_{ij}) \geq 0 \quad \forall (i,j) \in A_F, \quad \forall k. \tag{6i}$$

REMARK 3. Note that the reformulation RNOP(Ω) could possibly benefit further by the generation of RLT constraints of the type, for example, (2c) * $H_i \quad \forall i \in D$ for which some of the incident arcs are in A_2, along with (6g) and (6h) being then written for all $(i,j) \in A_F$ that are involved in such products. The cost-benefit analysis of including these and other higher order product constraints in RNOP(Ω) are open to computational investigations. □

LINEARIZATION PHASE

In the RLT linearization phase, a single new variable is simply substituted in RNOP(Ω) for each quadratic polynomial term that involves the product of some pair of the original problem variables. Specifically, we make the following substitutions of new variables to represent the corresponding nonlinear terms.

$$h_{ij}^+ = Q_{ij}H_i \text{ and } h_{ij}^- = Q_{ij}H_j \quad \forall (i,j) \in A_2, \text{ and} \quad \lambda_{ijk} = Q_{ij}x_{ijk} \quad \forall (i,j) \in A_F \quad \forall k, \tag{7}$$

where $x_{ijk} \equiv x_{jik}$. This produces a linear programming relaxation **RLT**[**NOP**(Ω)], say. Note that lower and upper bounds on the new variables that are implied by the polynomial terms they represent, along with bounds on the original variables, can be included for algorithmic purposes in solving RLT[NOP(Ω)]. Hence, solutions that are feasible to the original problem are always feasible to the linear relaxation, but not vice versa. Therefore, RLT[NOP(Ω)] yields valid lower bounds.

REMARK 4. Note that the h_{ij}^+ and h_{ij}^- variables are only included in the formulation when the flow direction is not known; that is, when $(i, j) \in A_2$ for which constraints (5d) become necessary. The remaining linearized variables λ are included whenever there is a possibility of a positive flow for the corresponding arc; that is, whenever $(i, j) \in A_F$. Notice also that if the relationships in (7) hold as an equality for *all* of the RLT variables, and the flows Q_{ij} coincide with some $\bar{Q}_{ij}$ used in (4) and (5) for each $(i, j) \in A_F$, then the LP relaxation solves the original problem exactly. □

The following two results summarize the key properties of RLT[NOP)Ω)]. The first of these results is evident from the foregoing discussion.

PROPOSITION 1. *For any hyperrectangle* Ω *bounding the flows* Q_{ij}, *we have* ν[RLT[NOP(Ω)]] $\leq \nu$[NOP(Ω)], *where* $\nu[\cdot]$ *denotes the objective function value at optimality for a given problem* $[\cdot]$. *Moreover, if* $(\bar{Q}, \bar{H}, \bar{x}, \bar{h}^+, \bar{h}^-, \bar{\lambda})$ *solves* RLT[NOP(Ω)], *and if* $(\bar{Q}, \bar{H}, \bar{x})$ *is feasible to* NOP*(Ω), then* $(\bar{Q}, \bar{H}, \bar{x})$ *solves* NOP*(Ω).* □

Proposition 2 below shows that if at any feasible solution to RLT[NOP(Ω)], some flow value Q_{ij} for a particular arc (i, j) at a branch-and-bound node is at one of its bounds in the Ω-hyperrectangle defining the node, then the relationships in (7) hold true for that variable. This property is essential in establishing the convergence of the proposed branch-and-bound algorithm.

PROPOSITION 2. *Let* $(Q, H, x, h^+, h^-, \lambda)$ *be any* feasible *solution to* RLT[NOP(Ω)] *for some defined hyperrectangle* Ω, *and suppose that an arc flow* Q_{ij} *satisfies* $Q_{ij} = Q_{ij}^L$ *or* $Q_{ij} = Q_{ij}^U$ *in this solution. Then the RLT-variable relationships in (7) hold true for all the variables that are associated with* Q_{ij} *in this solution.*

Proof. Suppose that $Q_{ij} = Q_{ij}^L$ in a feasible solution to RLT[NOP(Ω)]. In what follows, let us denote by $[\cdot]_L$ the linearized expression obtained upon substituting (7) into a corresponding nonlinear expression $[\cdot]$ appearing in Problem RNOP(Ω).

First, let us show that $h_{ij}^+ = Q_{ij}^L H_i$ and that $h_{ij}^- = Q_{ij}^L H_j$. Recall that h_{ij}^+ and h_{ij}^- exist only for $(i, j) \in A_2$, for which $Q_{ij}^L = 0$. Hence, we need to show that $h_{ij}^+ = h_{ij}^- = 0$ in this case. The first constraint in (6g) yields the pair of inequalities $[(H_i + E_i - H_{iL})Q_{ij}]_L \geq 0$ and $[(H_{iU} - H_i - E_i)Q_{ij}]_L \geq 0$, which respectively simplify to $h_{ij}^+ \geq Q_{ij}(H_{iL} - E_i) = 0$ and $h_{ij}^+ \leq Q_{ij}(H_{iU} - E_i) = 0$ when $Q_{ij} = Q_{ij}^L = 0$. This implies that $h_{ij}^+ = 0$. Similarly, using the second constraint in (6g), we obtain $h_{ij}^- = 0$.

Now, let us show that $\lambda_{ijk} = Q_{ij}x_{ijk} \equiv Q_{ij}^L x_{ijk} \quad \forall k$. The first constraint in (6i) yields upon linearization that

$$\lambda_{ijk} \geq x_{ijk}Q_{ij}^L \quad \forall k. \tag{8a}$$

Furthermore, (6f) gives $\sum_k [\lambda_{ijk} + X_{ijk} Q_{ij}] = L_{ij} Q_{ij}$. Hence, when $Q_{ij} = Q_{ij}^L$, using this equation along with (2g) itself (multiplied by the constant Q_{ij}^L) gives

$$\sum_k (\lambda_{ijk} + X_{ijk} Q_{ij}^L) = L_{ij} Q_{ij}^L$$
$$= \sum_k (x_{ijk} Q_{ij}^L + X_{ijk} Q_{ij}^L), \text{ i.e., } \sum_k \lambda_{ijk} = \sum_k x_{ijk} Q_{ij}^L. \tag{8b}$$

Equations (8a) and (8b) together imply that we must have

$$\lambda_{ijk} = x_{ijk} Q_{ij}^L \quad \forall k. \tag{9}$$

The arguments for the case $Q_{ij} = Q_{ij}^U$ follow identically, and this completes the proof. □

REMARK 5 (*Size of Problem* RLT[NOP(Ω)] *and its Solution).* Problem RLT[NOP(Ω)] has $2|S| + |D| + (9 + 2K)|A_1| + (17 + 2K)|A_2|$ structural constraints, aside from simple lower and upper bounds on the variables, and has $2|S| + |D| + (1 + 2K)|A_1| + (3 + 1.5K)|A_2|$ variables. (Here, we have used the fact that $|A_F| = |A_1| + |A_2|$ and that $|A_F, i < j| = |A_1| + 0.5|A_2|$.) Note that the size of the problem is more influenced by the number of links that do not have their flow directions determined, because of the additional restrictions (6g) and (6h), and the additional variables h_{ij}^+ and h_{ij}^- in this case. Moreover, due to the RLT relaxations used for (5), a potentially tighter representation results when flow directions are determined. Assuming for simplicity that all the original links in the network are *undirected*, then in the worst case, when none of the links have their flow directions determined, we have, $|A_1| = 0$ and $|A_2| = |A|$. Hence, the number of constraints is $2|S| + |D| + (17 + 2K)|A|$, and the number of variables is $2|S| + |D| + (3 + 1.5K)|A|$. In the best case, when all the links have their flow directions determined, $|A_1| = 0.5|A|$ and $|A_2| = 0$, and the number of constraints is then given by $2|S| + |D| + (4.5 + K)|A|$, and the number of variables is $2|S| + |D| + (0.5 + K)|A|$.

For example, in the test problem of Section 5, where $|S| = 1$, $|D| = 6$, $|A| = 14$, and $K = 5$, we obtain 386 constraints and 155 variables in the worst case. In the best case, we obtain 141 constraints and 85 variables. About 63% of the variables are eliminated by resolving the directions of flow. □

4. Branch-and-Bound Algorithm

We now embed RLT[NOP(Ω)] in a branch-and-bound procedure to solve Problem NOP globally to within any specified $\varepsilon > 0$, or percentage, tolerance. The discussion in this section is based on the development in Sherali and Tuncbilek (1992) and Sherali (1996) where general polynomial programs are addressed, but differs in our treatment of constraints (2d) via the representation given in (5), and also differs in the critical design of a partitioning strategy. (Also, see Horst *et al.* (1995) for a general discussion on branch-and-bound techniques.) Our approach is to perform suitable rectangular partitions on the subset of variables defined by the hyperrectangle $\Omega = \{Q : Q^L \leq Q \leq Q^U\}$. Let the hyperrectangle associated with node t of the branch-and-bound enumeration tree be denoted by Ω^t, where $\Omega^t \equiv \{Q : Q^{L_t} \leq Q \leq Q^{U_t}\}$. Then ν[RLT[NOP (Ω^t)]] yields a *lower bound* for the node subproblem NOP(Ω^t). (Note that we will continue to assume that if $Q_{ij}^L = Q_{ij}^U$ for any $(i, j) \in A$, then Q_{ij} is fixed at this common value in the problem and is no longer treated as a variable in the computations.) In particular, if $(\bar{Q}, \bar{H}, \bar{x}, \bar{h}^+, \bar{h}^-, \bar{\lambda})$ solves RLT[NOP(Ω^t)] and the corresponding solution $(\bar{Q}, \bar{H}, \bar{x})$ is feasible to NOP(Ω^t), then by Proposition 1, $(\bar{Q}, \bar{H}, \bar{x})$ solves NOP(Ω^t), and being feasible to NOP(Ω), the value ν[NOP(Ω^t)]$\equiv$ ν[RLT[NOP(Ω^t)]] provides an *upper bound* for Problem NOP(Ω). Hence, we have a candidate for possibly updating the incumbent solution (Q^*, H^*, x^*) and its value ν^* for Problem NOP(Ω).

In case $(\bar{Q}, \bar{H}, \bar{x})$ is not feasible to NOP(Ω), we can possibly determine an improved *upper bound* ν^* for NOP(Ω) by applying some heuristic to the lower bounding solution. For example, starting from $\bar{Q}$, we can apply the heuristic of Alperovits and Shamir (1977), or any other relatively quick and effective local optimization scheme cited in Section 1. (Note that although these heuristics can be complex and involve nondifferentiable optimization subproblems [see Eiger *et al.* (1994) for example], any simplified version of such heuristics can be run for only a restricted number of iterations in this context.) For our computations, we used the following simple heuristic. Given $\bar{x}$ and $\bar{Q}$, we examine each link and determine for this link a (continuous) diameter segment that would yield the same hydraulic head loss as in the current solution. From this, we can then determine (up to) two adjacent standard pipe diameters that contain this computed continuous diameter value in between them, along with the lengths of the corresponding pipe segments to yield the same head loss. Let $\hat{x}$ represent the obtained network design. Fixing $x = \hat{x}$, we then use the method recommended by Walski (1984) to solve a system of nonlinear equations that yields the accompanying flows and pressure heads. If this is feasible, then we update the incumbent value. Otherwise, if the resulting solution is infeasible, but its objective value is lesser than the incumbent value by some amount Δ, we increase the cost of the design, in turn, by 50%, 70%, and then 90% of this difference Δ by proportionately increasing the effective pipe diameter of each link to the value that increases the cost by this percentage amount, thereby finding a revised network design $\hat{x}$ to which Walski's procedure is applied. Since this successively costlier design has large diameter pipes that result in smaller head losses, we have an increasingly greater likelyhood of finding an improved feasible solution.

Now, if ν[RLT[NOP(Ω^t)]]$\geq$ ν^*, we can fathom node t. Hence, at any stage r of the branch-and-bound algorithm, we have a set of non-fathomed or *active nodes* denoted by T_r. Given this, we *select an active node* t^* in T_r that has the least objective function value for the corresponding relaxation RLT[NOP$(\cdot)$] (breaking ties arbitrarily). That is, we select $t^* \in$ argmin$\{\nu$[RLT[NOP(Ω^t)]] $: t \in T_r\}$. Next, we partition the hyperrectangle associated with this node t^* into two sub-hyperrectangles based on the following *branching variable selection strategy*. The design of this strategy is not only important from the viewpoint of computational efficiency, but is also critical in ensuring the theoretical convergence of the overall procedure. In this strategy, given $(\bar{Q}, \bar{H}, \bar{x})$ that has been obtained as (part of) an optimal solution to RLT[NOP(Ω^t)], we define

$$\delta_{ij} = (Q_{ij}^{U_t} - Q_{ij}^{L_t}) \min\{\bar{Q}_{ij} - Q_{ij}^{L_t}, Q_{ij}^{U_t} - \bar{Q}_{ij}\} \quad \forall (i,j) \in A_F. \tag{10a}$$

The branching variable choice is then given by

$$(p,q) = \underset{(i,j)\in A_F}{\text{argmax}}\{\delta_{ij}\}, \tag{10b}$$

and we accordingly partition the current interval $[Q_{pq}^{L_t}, Q_{pq}^{U_t}]$ for Q_{pq} into the subintervals

$$[Q_{pq}^{L_t}, \bar{Q}_{pq}] \text{ and } [\bar{Q}_{pq}, Q_{pq}^{U_t}], \tag{10c}$$

noting that $Q_{pq}^{L_t} < \bar{Q}_{pq} < Q_{pq}^{U^t}$ when $\delta_{pq} > 0$. The following lemma asserts a simple, but crucial, fact that will be useful in establishing the convergence of the proposed algorithm, and motivates the design of the above branching variable selection strategy.

PROPOSITION 3. *With (p,q) selected as in (10), if $\delta_{pq} = 0$, then the (partial) optimum solution $(\bar{Q}, \bar{H}, \bar{x})$ to* RLT[NOP(Ω^t)] *solves* NOP(Ω^t), *yielding the same objective value. Hence, the corresponding node t can be fathomed, after updating the incumbent feasible solution, if necessary.*

Proof. Since $\delta_{pq} = 0$, (10) implies that $\bar{Q}_{ij} = Q_{ij}^{L_t}$ or $\bar{Q}_{ij} = Q_{ij}^{U^t}$ (or both) for each $(i,j) \in A_F$. By Proposition 2, the relationship (7) holds true for the solution of problem RLT[NOP(Ω^t)]. Moreover, from (4) and (5), since the approximations to (2d) hold exactly at the end points of the flow intervals, the solution $(\bar{Q}, \bar{H}, \bar{x})$ satisfies the constraints (2d) $\forall (i,j) \in A$ for Problem NOP(Ω^t). All the remaining constraints in NOP(Ω^t) are satisfied explicitly by inclusion in the formulation of Problem RLT[NOP(Ω^t)]. Hence, the optimal solution to RLT[NOP(Ω^t)] is feasible to NOP(Ω^t) and, by Proposition 1, solves NOP(Ω^t). □

A formal statement of the proposed branch-and-bound algorithm is given below.

Branch-and-Bound Algorithm

Initialization Step. Initialize the incumbent solution $(Q^*, H^*, x^*) = \emptyset$ and $\nu^* = \infty$. (Practically, if some feasible solution is known, this could be used as an initial solution.) Apply the procedure of Remark 2 to determine valid bounds on the flow variables, along with a possibly improved incumbent solution (Q^*, H^*, x^*) of objective value ν^*. Set the stage counter $r = 1$, and let $T_1 = \{1\}$. Denote $\Omega^{rt} = \Omega^{1,1} \equiv \Omega$ as the initial hyperrectangle that defines Problem NOP(Ω). Solve RLT[NOP$(\Omega^{1,1})$] to obtain an optimal solution of objective function value $LB_{1,1} \equiv \nu$[RLT[NOP(Ω)]] and determine a branching variable by using Equation (10). If $\delta_{pq} = 0$, or if $\nu^* = LB_{1,1}$, then stop; by Proposition 3, this solution solves the original Problem NOP(Ω). Otherwise, apply a suitable heuristic (as described above) to the optimum obtained for Problem RLT[NOP(Ω^t)] in order to possibly improve the incumbent solution. Set $t^* = 1$, and proceed to Step 1.

Step 1. Partitioning Step (Stage r, $r \geq 1$). Having the active node (r, t^*) to be partitioned, and given the choice (p, q) for the branching variable as determined by (10), partition this node into two sub-nodes associated with the two sub-hyperrectangles Ω^{r,t_1} and Ω^{r,t_2} that are identical to $\Omega^{r,t}$, except that the two respective interval restrictions on Q_{pq} are given by (10c) corresponding to $t = t^*$ at stage r. (See Remark 6 below for a pertinent suggestion on this partitioning strategy.) Update $T_r \leftarrow (T_r - \{t^*\}) \cup \{t_1, t_2\}$, and proceed to Step 2.

Step 2. Bounding Step. Solve the linear program RLT[NOP(Ω^{r,t_1})]. If this problem is infeasible, then fathom the corresponding node at Step 3. Otherwise, find an optimal solution, denote its objective function value by $LB_{r,t_1} = \nu$[RLT[NOP(Ω^{r,t_1})]], and using this optimal solution in (10), determine the corresponding branching variable index (p, q). If $\delta_{pq} = 0$, then by Proposition 3, this solution solves the node subproblem NOP(Ω^{r,t_1}). In this case, if $\nu^* > LB_{r,t_1} \equiv \nu$[RLT[NOP$(\Omega^{r,t_1})$]], then update the incumbent solution (Q^*, H^*, x^*) and its value ν^* accordingly. Else, we have $\delta_{pq} > 0$, and so, store the branching variable index (p, q) to be possibly used later. Also, apply a suitable heuristic (as described above) to the optimum for Problem RLT[NOP(Ω^{r,t_1})] in order to possibly improve the incumbent solution. Repeat Step 2 after replacing t_1 by t_2, and then proceed to Step 3.

Step 3. Fathoming Step. Fathom any nonimproving nodes by setting $T_{r+1} = T_r - \{t \in T_r : LB_{r,t} \geq \nu^*\}$. If $T_{r+1} = \emptyset$, then stop. Otherwise, update $\Omega^{r+1,t} = \Omega^{r,t}$ and $LB_{r+1,t} = LB_{r,t}$ for all $t \in T_{r+1}$. Increment r by 1, and proceed to Step 4.

Step 4. Node Selection Step. Select an active node (r, t^*), where $t^* \in \text{argmin}\{LB_{r,t} : t \in T_r\}$ is associated with the least lower bound $LB_r \equiv LB_{r,t}$, over the active nodes at stage r. Return to Step 1.

REMARK 6 (*A Practical Partitioning Consideration).* In Remark 5, we emphasized the benefit of having flow directions determined with regard to the size and tightness of the RLT relaxation. Accordingly, at Step 1 of the foregoing procedure, whenever an arc $(p, q) \in A_2$ is selected for partitioning for the first time, the partitioning can be performed by letting $Q_{pq} \in [0, Q_{pq}^{U_t}]$ and $Q_{qp} \equiv 0$ at one node, and letting $Q_{pq} = 0$ and $Q_{qp} \in [0, Q_{qp}^{U_t}]$ at the other node, hence determining the direction of flow for the link (p, q) at each subnode. (Note that this would not affect the theoretical convergence of the algorithm.) □

We now address the convergence of the proposed algorithm.

PROPOSITION 4 (*Convergence Result). The above branch-and-bound algorithm either terminates finitely with the incumbent solution being optimal to* NOP(Ω)*, or else, an infinite sequence of stages is generated. In the latter case, along any infinite branch of the branch-and-bound tree, any accumulation point of the sequence of solutions* (Q, H, x) *generated via the optimal linear programming solutions obtained for the relaxations* RLT[NOP(Ω^t)] *corresponding to the nodes of this branch, solves Problem* NOP(Ω).

Proof. The case of finite termination is clear. Hence, suppose that an infinite sequence of stages is generated. As in Sherali and Tuncbilek (1992), along any infinite branch of the branch-and-bound tree, we have a nested subsequence of partitions $\{\Omega^{r,t(r)}\}_R$ indexed by $r \in R$, where

$$t(r) \equiv t^* \in \text{argmin}\{\nu[\text{RLT[NOP}(\Omega^{r,t})]] : t \in T_r\} \text{ for each } r \in R,$$

and where each partition at stage $r \in R$ corresponds to the same branching variable (p, q), say. Moreover, this subsequence is such that the corresponding sequence of solutions to RLT[NOP$(\Omega^{r,t(r)})$] converges to some solution $(\bar{Q}, \bar{H}, \bar{x})$, and that $\{\Omega^{r,t(r)}\} \to \bar{\Omega} \equiv \{Q : \bar{Q}^L \leq Q \leq \bar{Q}^U\}$, where $\bar{Q}_{pq} = \bar{Q}_{pq}^L$ or $\bar{Q}_{pq} = \bar{Q}_{pq}^U$ holds true. We must now show that $(\bar{Q}, \bar{H}, \bar{x})$ solves Problem NOP(Ω).

Since $\bar{Q}_{pq} = \bar{Q}_{pq}^L$ or $\bar{Q}_{pq} = \bar{Q}_{pq}^U$ holds true as above, we have that $\delta_{pq} \to 0$ as $r \to \infty$, $r \in R$. Hence, by (10a) and (10b), we have that $\delta_{ij} \to 0$ as $r \to \infty$, $r \in R$, and so, $\bar{Q}_{ij} = \bar{Q}_{ij}^L$ or $\bar{Q}_{ij} = \bar{Q}_{ij}^U \quad \forall (i, j) \in A_F$ as well. By Proposition 3, then, it follows that $(\bar{Q}, \bar{H}, \bar{x})$ is feasible to NOP$(\bar{\Omega})$, and is therefore feasible to NOP(Ω), because $\bar{\Omega} \subset \Omega$. Therefore, $(c \cdot \bar{x} + c_s \cdot \bar{H}_s)$ serves as an upper bound for NOP(Ω). But ν[RLT[NOP$(\Omega^{r,t(r)})$]]$_R$ provides a sequence of global lower bounds for Problem NOP(Ω), and this sequence of values also approaches $(c \cdot \bar{x} + c_s \cdot \bar{H}_s)$. Hence, $(c \cdot \bar{x} + c_s \cdot \bar{H}_s)$ serves as both an upper and lower bound on ν[NOP(Ω)], and so, $(\bar{Q}, \bar{H}, \bar{x})$ solves Problem NOP(Ω). This completes the proof. □

5. Computational Experience on a Standard Test Problem

In this section, we apply the proposed branch-and-bound algorithm to Alperovits and Shamir's (AS(1977)) single source test problem, and to several of its variants. Actually, there exists some confusion in the literature in reference to this test problem, since the original paper AS(1977) considers only a restricted set of pipe diameters in stating the test problem, but the solution given includes a 4" pipe diameter for one of the links (5, 7) (see Figure 1) that is not part of the printed data. Moreover, several other authors have attempted to solve this problem by permitting

Table 1. Test problem node data

Node	Elevation E_i (m)	Minimum and Maximum Pressure Bounds on H_i (m)	Supply or Demand (m^3/hr)	H_{iL} (m)	H_{iU} (m)
1	210	—	1120	N/A	N/A
2	150	30, 60	−100	180	210
3	160	30, 50	−100	190	210
4	155	30, 55	−120	185	210
5	150	30, 60	−270	180	210
6	165	30, 45	−330	195	210
7	160	30, 50	−200	190	210

certain smaller pipe diameters, including a 1" diameter in particular, for links (4, 5) and (5, 7). Hence, for the sake of illustration and comparison, we solve several variants of this problem. First, in Section 5.1, we consider the AS(1977) problem including the 4" pipe for link (5, 7) and provide comparisons with the solution given in AS(1977). Next, in Section 5.2, we solve the original data problem of AS(1977) for the sake of providing another test case for which we have obtained a global optimum. Thereafter, in Section 5.3, we solve the problem of AS(1977) permitting the smaller diameters (1" in particular) for the links (4,5) and (5,7), and compare our results with those obtained by Eiger *et al.* (1994) and Loganathan *et al.* (1995). Finally, in Section 5.4, we consider the same data as in Section 5 except that the head loss constraint (1c) is taken as (1b), corresponding to rough flow conditions.

5.1. AS (1977) TEST PROBLEM INCLUDING A 4" PIPE FOR LINK (5, 7)

Data for this problem are presented in Tables 1 and 2, and in Figure 1, along with selected bounds on the head variables. Note that in our run, no *a priori* bounds on the flows were assumed, and that the initial bounds were found using the provably valid scheme described in Remark 2. The original test problem has a rational exponent of 1.852 for Q in the head-loss constraints, and so, the form of (1a) was used in (1c) with $e = 1.852$. The algorithm was implemented on a SUN SPARC 10 Unix workstation, using the CPLEX callable library to solve the linear programming subproblems. The RLT computer code was written in SUN FORTRAN 77, while the CPLEX code provided by the CPLEX Corporation is written in C.

For this problem instance, our branch-and-bound algorithm enumerated 57 nodes (7.5 minutes of cpu time) to optimally solve the problem. The initial (node zero) lower bound was \$210,087, and the final global lower and upper bounds were \$426,403 and \$426,404, respectively. Hence, this problem has been solved for the first time to within \$1 of optimality. The solution obtained is given in Table 3. This

Table 2. Test problem arc and cost data

Link Index	Acrs	Length (m)	C_{HWN}	Allowable Pipe Diameters(*in*)
1	(1,2)	1000	130	12, 14, 16, 18, 20
2	(2,3)	1000	130	6, 8, 10, 12, 14
3	(2,4)	1000	130	10, 12, 14, 16, 18
4	(4,5), (5,4)	1000	130	3, 4, 6, 8, 10
5	(4,6), (6,4)	1000	130	10, 12, 14, 16, 18
6	(6,7), (7,6)	1000	130	8, 10, 12, 14, 16
7	(3,5), (5,3)	1000	130	6, 8, 10, 12, 14, 16
8	(5,7), (7,5)	1000	130	4, 6, 8, 10, 12

COST DATA

Pipe diameter (*in*)	1	2	3	4	6	8	10	12	14	16	18	20	22	24
Cost ($/m)	2	5	8	11	16	23	32	50	60	90	130	170	300	550

Table 3. Optimal solutions obtained for the test problem of Section 5.1

Pipe Section # (i, j)	(x^*) Sections Having Length (m) of Diameter (*in*)		(Q^*) Flow (m^3/hr)
1(1,2)	1000.0	18"	1120.00
2(2,3)	1000.0	10"	326.21
3(2,4)	1000.0	16"	693.79
4(4,5)	1000.0	3"	17.26
5(4,6)	12.4	16"	556.53
	987.6	16"	
6(6,7)	688.3	10"	226.53
	311.7	12"	
7(3,5)	426.2	8"	226.21
	573.8	10"	
8(7,5)	1000.0	4"	26.53

solution yields an improvement of 11% over the previous best solution reported by Alperovits and Shamir having an objective value of $479,525. Moreover, the global lower and upper bounds had a smaller than 5% gap at node 31, and a smaller than 1% gap at node 42. Hence, for practical purposes, an earlier termination could have been effected.

5.2. AS (1977) Test Problem Excluding the 4" Pipe for Link (5,7)

In this variant of the AS(1977) test problem presented in Section 5.1, we assumed that the permissible diameter pipes for link (5,7) were 6", 8", 10", 12", and 14"

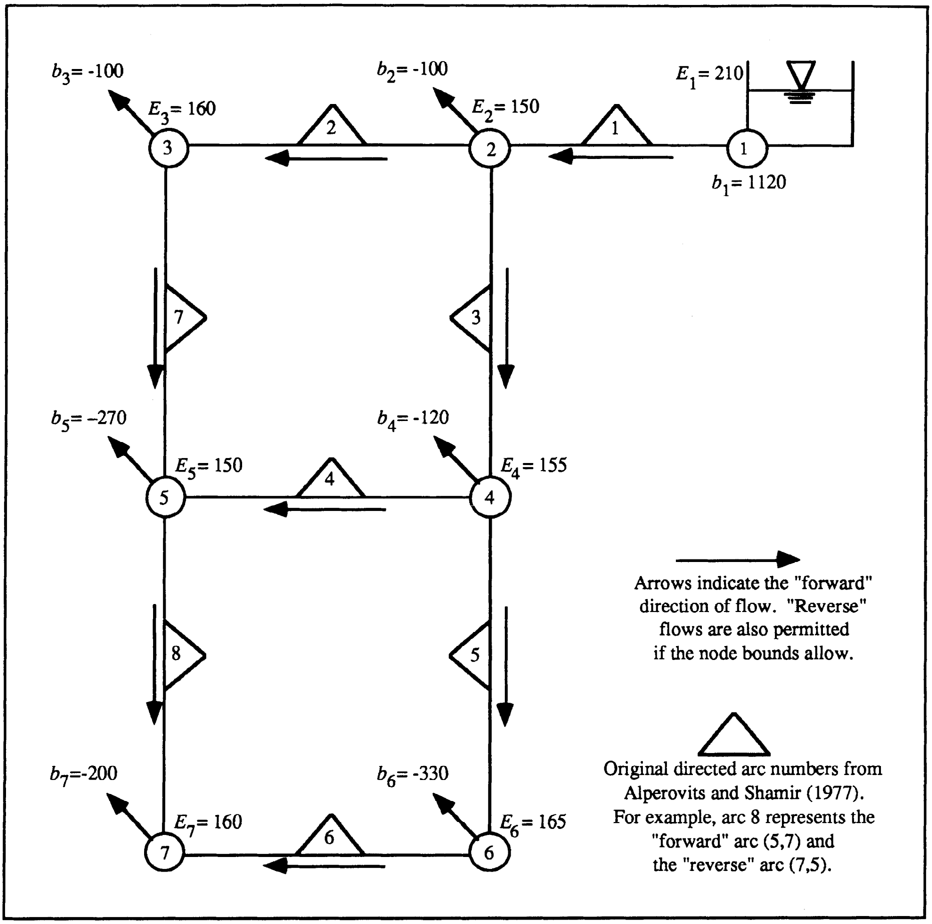

Fig. 1.

(in lieu of the desired 4" pipe for this link). Again, no *a priori* flow bounds were assumed, and the initial bounds on the flows were computed using the method described in Remark 2. The initial (node zero) lower bound was \$215,350, and the algorithm enumerated 57 branch-and-bound nodes in 7.1 minutes of cpu time, terminating with exactly matching global lower and upper bounds of \$436,684. The corresponding solution obtained is given in Table 4. In this run, the gap between the global lower and upper bounds was lesser than 5% at node 34, and was lesser than 1% at node 46.

5.3. AS (1977) TEST PROBLEM PERMITTING SMALLER DIAMETER PIPES FOR LINKS (4,6) AND (5,7)

We now solve a variant of the AS(1977) test problem that permits the 1" pipe diameter for the links (4,5) and (5,7), as used by several other authors. The specific

Table 4. Optimal solutions obtained for the test problem of Section 5.2

Pipe Section # (i, j)	(x^*) Sections Having Length (m)	of Diameter (*in*)	(Q^*) Flow (m^3/hr)
1(1,2)	1000.0	18"	1120.00
2(2,3)	1000.0	14"	447.61
3(2,4)	214.2	14"	572.39
	785.8	16"	
4(4,5)	999.9	3"	9.47
	0.1	4"	
5(4,6)	1000.0	14"	442.92
6(6,7)	1000.0	8"	112.92
7(3,5)	49.4	10"	347.61
	950.6	12"	
8(5,7)	1000.0	8"	87.08

diameters we permitted were 1, 2, 3, 4, and 6 inches for both these links. The algorithm enumerated 49 nodes in 5.7 minutes of cpu time, terminating with global lower and upper bounds of \$403,385 and \$403,386, respectively. The corresponding solution obtained is given in Table 5, and is guaranteed to be within \$1 of optimality. Again, note that theoretically justified flow bounds were derived for the overall problem at node zero using the method of Remark 2. The effort for deriving these bounds is only 35 cpu seconds of the total 5.7 cpu minutes, but this step is crucial for solving the problem. For example, without this step, and using only logically implied flow bounds, after enumerating 50 nodes, the global lower bound was only \$268,948. In this run, the initial lower bound at node zero after the flow bounds were determined turned out to be \$194,478. The gap between the global lower and upper bounds was reduced to under 5% by node 32, and to under 1% by node 35, indicating that the procedure could have terminated earlier (using 35% lesser effort) with a near optimal (1% optimality tolerance) solution.

In comparison, Eiger *et al.* (1994) use certain (unspecified) heuristically determined flow bounds, and solve this problem up to 0.4% use certain (unspecified) heuristically determined flow bounds, and solve this problem up to 0.4% of optimality in 0.9 minutes on a SUN Sparc 4 workstation. Their procedure enumerated 204 nodes, and terminated with a global lower bound of \$400,743 and an upper bound of \$402,352 (compared with our global lower and upper bounds of \$403,385 and \$403,386, respectively). The discrepancy in their upper bound and our global lower bound arises here because the “optimal solution” reported by Eiger *et al.* is actually somewhat infeasible. On the other hand, using their *heuristic* procedure, Loganathan *et al.* (1995) have reported the best known solution to this problem, prior to the present work, of objective value \$403,657 with no reported lower bound

Table 5. Optimal solutions for the test problem of Section 5.3

Test Problem of Section 5.3				Test Problem of Section 5.4			
Pipe Section # (i,j)	(x^*) Sections Having Length (m)	of Diameter (*in*)	(Q^*) Flow (m^3/hr)	Pipe Section # (i,j)	(x^*) Sections Having Length (m)	of Diameter (*in*)	(Q^*) Flow (m^3/hr)
1(1,2)	1000.0	18"	1120.0	1(1,2)	1000.0	20"	1120.0
2(2,3)	795.4	10"	368.33	2(2,3)	628.6	10"	367.63
	204.6	12"			371.4	12"	
3(2,4)	1000.0	16"	651.67	3(2,4)	827.7	16"	652.37
					172.3	18"	
4(4,5)	1000.0	1"	0.98	4(4,5)	1000.0	1"	1.37
5(4,6)	310.3	14"	530.69	5(4,6)	1000.0	16"	531.00
	689.7	16"					
6(6,7)	11.1	8"	200.69	6(6,7)	862.3	10"	201.00
	988.9	10"			137.7	12"	
7(3,5)	98.5	8"	268.33	7(3,5)	18.7	8"	267.63
	901.5	10"			981.3	10"	
8(7,5)	1000.0	1"	0.69	8(7,5)	1000.0	1"	1.00

to assess the quality of this solution. We now know that their solution was within 0.07% of optimality.

5.4. AS(1977) TEST PROBLEM OF SECTION 5.3 UNDER ROUGH FLOW CONDITIONS.

We now consider the same test problem as described in Section 5.3, except that we use the integer exponent of $e = 2$ on the flow variables in the head loss constraint as in (1b), corresponding to rough flow conditions. (A value of $\tilde{Q}_{ij} = 100\, m^3/\text{hr} \quad \forall (i,j)$ was used in (1b) for our computations.) As for the previous cases, the procedure of Remark 2 was used to ascertain provable initial flow bounds. This yielded an initial global lower bound on the problem of $213,531. After enumerating 39 nodes in 9.30 minutes of cpu time, the final global lower and upper bounds obtained upon termination were both $465,887, hence solving this problem exactly. Table 5 gives the results obtained. In this run, the gap between the global lower and upper bounds was less than 5% after enumerating 29 nodes, while this gap was less than 1% after enumerating 31 nodes.

Note that the optimal cost of $465,887 is significantly larger than that for the solution obtained in Section 5.3 ($403,386) corresponding to the rational exponent case. In fact, the exponent of 2 corresponding to rough flow conditions in the Hazen–Williams equation will, in general, produce more expensive solutions since it results in greater head losses than does the exponent of 1.852 under smooth flow conditions. The ensuing compensation between flow rates and increased pipe diameters in order to maintain adequate pressure heads, naturally produces a more expensive design using the larger (integer) exponent.

6. Summary and Conclusions

In this paper, we have presented a model for the single-stage pipe network design problem, the solution of which constitutes a principal step in an overall process of designing a water distribution system. For this model, we have developed a global optimization approach, and have presented some promising, preliminary computational results on a particular test problem from the literature, reporting for the first time true global optimal solutions for several variants of this test problem that can be useful for future research.

In conclusion, we mention six possible major improvements that can be made to the algorithm that has been proposed in this paper. First, we can investigate the generation of additional classes of constraints such as the ones discussed in Remark 3. Although not required to guarantee theoretical convergence, additional constraints of this type can serve to tighten the relaxations generated sufficiently to possibly yield a cost-benefit advantage.

Second, better heuristics need to be developed (or implemented) that can quickly find good quality feasible solutions when starting from an infeasible solution obtained via a lower bounding problem solved at a node in the branch-and-bound tree. We only just began to investigate this area, and the heuristic we implemented was a simple perturbation local-search scheme. However, it is encouraging that this scheme was able to generate feasible solutions of good enough quality to quickly narrow the lower-upper bound gap.

Third, several alternatives admissible branching strategies are evident (see Sherali (1996) for example) that can also lead to theoretical convergence. Such strategies could be investigated and tested for computational effectiveness.

Fourth, there exist several “preprocessing” range-reduction strategies that can be used to further tighten the bounding hyperrectangle of flows at any node of the branch-and-bound tree. Such range-reduction schemes have been successfully employed by Thakur (1990), Hansen *et al.* (1991), Ryoo and Sahinidis (1994), Sherali and Tuncbilek (1995), Shectman and Sahinidis (1994), and Lamar (1995), among others.

Fifth, we could employ nonlinear outer approximations to the feasible region that generate nonlinear, but convex, branch-and-bound node subproblems. In their *Reformulation-Convexification Technique*, Sherali and Tuncbilek (1995) demonstrate how such constraints can significantly tighten the relaxation, without increasing the ensuing effort required to solve it.

Finally, given the size of the relaxations generated (see Remark 5), for large-scale problems, it becomes imperative to use some suitable (deflected) subgradient technique applied to a Lagrangian dual formulation of the lower bounding problem RLT[NOP(Ω)]. (A scaling of the problem prior to applying RLT, e.g., scaling all flow variables onto the interval [0,1], can help in the rate of convergence in this process.) As demonstrated in Sherali and Tuncbilek (1995), for example, this approach can not only significantly reduce the computational burden of coping with the size of the relaxations produced by RLT, but can also permit the handling of simple nonlinear, convex, variable bounding constraints with negligible additional effort.

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